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Averaging the k largest distances among n: k-centra in Banach spaces

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Abstract

Given a Banach space X let $A \subset X$ containing at least k points. In location theory, reliability analysis, and theoretical computer science, it is useful to minimize the sum of distances from the k furthest points of A: this problem has received some attention for X a finite metric space (a network), see, e.g., [Discrete Appl. Math. 109 (2001) 293]; in the case $X = E^n$, $k = 2$ or 3, and A compact some results have been given in [Math. Notes 59 (1996) 507]; also, in the field of theoretical computer science it has been considered in [T. Tokuyama, Minimax parametric optimization problems in multi-dimensional parametric searching, in: Proc. 33rd Annu. ACM Symp. on Theory of Computing, 2001, pp. 75–84]. Here we study the above problem for a finite set $A \subset X$, generalizing—among others things—the results in [Math. Notes 59 (1996) 507].

1. Introduction

Let X be a Banach space; let $A = \{a_1, \dots, a_n\} \subset X$, $n \geqslant 3$, $a_i \neq a_j$ for $i \neq j$, a finite set whose cardinality will be denoted by $\#A$. Also, we denote by $\delta(A)$ the diameter of A.

Given $x \in X$, let $\sigma(x) = (\sigma_1(x), \dots, \sigma_n(x))$ be an ordering of the elements of $\{1, 2, \dots, n\}$ such that $\|x - a_{\sigma_1(x)}\| \geqslant \|x - a_{\sigma_2(x)}\| \geqslant \cdots \geqslant \|x - a_{\sigma_n(x)}\|$.

Given an integer k, $1 \leqslant k \leqslant n$, we set:

$$r_k(A, x) = \frac{1}{k} \sum_{i=1}^{k} \|x - a_{\sigma_i(x)}\| \quad \text{and} \quad r_k(A) = \inf_{x \in X} r_k(A, x).$$

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Clearly, $r_1(A)$ is the Chebyshev radius of A, that we shall also denote by $r(A)$, while $r_n(A)$ is the minimum average of distances from the points of A, usually denoted by $\mu(A)$. (We also use this notation when referring to others' results.) A point x (when it exists) such that $r_k(A,x) = r_k(A)$ will be called a *k-centrum* of A.

In particular, a 1-centrum of A is a (Chebyshev) center; an n-centrum of A is a median (or Fermat point). The term k-centrum was coined in the early seventies [15] to refer to the minimization of the function $r_k(A,x)$ when X is a finite metric space. The reader should notice that this term (k-centrum) differs from *n-center* as it is used in recent papers. In the latter, n-center means center or median for n-point sets or n-flat of a given finite set.

In this paper, we study the functions $r_k(A,x)$ and the k-centra; these problems, apart from some results given in [23], have been also considered in [11,15–17] from an algorithmic point of view. The interested reader can also find different applications of these functions in different areas of applied mathematics as reliability: optimization of systems k-out-of-n [1]; location analysis [13] or in decision theory [22], among others.

2. Preliminary results

We start with a simple remark; clearly, given a finite set $A = \{a_1, \dots, a_n\}$, for any $x \in X$ we have

$$r_1(A,x) \geqslant r_2(A,x) \geqslant \cdots \geqslant r_n(A,x).$$

From this we have the following remark.

Remark 2.1. For any A we have

$$r(A) \geqslant r_2(A) \geqslant \cdots \geqslant r_{n-1}(A) \geqslant \mu(A). \tag{1}$$

Remark 2.2. We can also give estimates in the "opposite" sense. Let $1 \leqslant k \leqslant j \leqslant n$. Given any $A = \{a_1, \dots, a_n\}$, for every $x \in X$ we have $kr_k(A,x) = \sum_{i=1}^{k} \|x - a_{\sigma_i(x)}\| \leqslant \sum_{i=1}^{j} \|x - a_{\sigma_i(x)}\| = jr_j(A,x)$; taking infimum on x, we obtain

$$kr_k(A) \leqslant jr_j(A). \tag{2}$$

A better estimate is the following (whose proof is almost trivial) proposition.

Proposition 2.1. *Given $A = \{a_1, \dots, a_n\}$, let $n \geqslant 2h$ with h an integer $1 \leqslant h \leqslant n/2$. If i, j is a pair of indexes such that $\|a_i - a_j\| = \delta(A)$, set $A_1 = A \setminus \{a_i, a_j\}$; then let i_1, j_1 be indexes such that $a_{i_1}, a_{j_1} \in A_1$ and $\|a_{i_1} - a_{j_1}\| = \delta(A_1)$; then define $A_2 = A_1 \setminus \{a_{i_1}, a_{j_1}\}$. Proceeding in this way, we obtain*

$$2hr_{2h}(A) \geqslant \delta(A) + \delta(A_1) + \delta(A_2) + \cdots + \delta(A_{h-1}). \tag{3}$$

The next result gives us some structural properties of the $r_k(A,x)$ function. They are direct consequences of basic properties of the norm in X and thus, its proof is left out.

Proposition 2.2. *Let $A = \{a_1, \ldots, a_n\}$ and let k be an integer $1 \leqslant k \leqslant n$; then the function $r_k(A, x)$ $(x \in X)$ is 1-Lipschitz continuous and convex. Moreover, if X is strictly convex, $r_k(A, x)$ is strictly convex outside lines containing at least k points of A.*

Given A, let for $\varepsilon \geqslant 0$ and $1 \leqslant k \leqslant n = \#A$,

$$s_k(A, \varepsilon) = \{x \in X\colon\ r_k(A, x) \leqslant r_k(A) + \varepsilon\}. \tag{4}$$

According to Proposition 2.2, the sets $s_k(A, \varepsilon)$ are always closed and convex. Also, in a dual space, the functions $x \to \|x - a\|$ are weak*-lower semicontinuous, so the sets $s_k(A, \varepsilon)$ are bounded, w*-closed, and w*-compact. Therefore, the (possibly empty) set

$$s_k(A) = \bigcap_{\varepsilon > 0} s_k(A, \varepsilon) \tag{5}$$

is always closed, bounded, and convex, and its elements are the k-centra of A, i.e., the points x such that $r_k(A, x) = r_k(A)$.

By standard w*-compactness arguments we obtain the following proposition.

Proposition 2.3. *If X is a dual space (in particular, if X is reflexive), then $s_k(A) \neq \emptyset$ for any finite set A and any k between 1 and $\#A$.*

Remark 2.3. The above result is true, for example, if $X = l_\infty$. Also, the same result holds if X is norm-one complemented in X^{**}. The proof in the case of existence of norm-one projection is simple (and obtains following the line of proofs in [19]). General results of this type have been given in [19].

Next result shows that also other spaces have the same properties.

Theorem 2.1. *If $X = c_0$, then for every $A = \{a_1, \ldots, a_n\}$ and $1 \leqslant k \leqslant n$ we have $s_k(A) \neq \emptyset$.*

Proof. We may consider A as a subset of l^∞. Since l^∞ is a dual space, there exists $x = (x^{(1)}, x^{(2)}, \ldots, x^{(n)}, \ldots) \in l^\infty$ such that $r_k(A, x) = \inf\{r_k(A, y)\colon\ y \in l^\infty\}$. Since A is in c_0 there exists an index h such that $|a_i^{(j)}| \leqslant \|x - a_{\sigma_k(x)}\|$, for all $j > h$ and $i = 1, \ldots, n$. Then, $x_0 = (x^{(1)}, \ldots, x^{(h)}, 0, \ldots, 0, \ldots) \in c_0$ and

$$\|x_0 - a_i\| \leqslant \sup\{\sup\{|a_i^{(j)}|\colon\ j > h\}, \sup\{|x^{(j)} - a_i^{(j)}|\colon\ j \leqslant h\}\} \leqslant \|x - a_{\sigma_k(x)}\|,$$

for $i = 1, \ldots, n$. Hence, $r_k(A, x_0) \leqslant \|x - a_{\sigma_k(x)}\| \leqslant r_k(A, x) = r_k(A)$ and so $r_k(A, x_0) = r_k(A)$. □

Remark 2.4. There are spaces where for some finite sets, centers and/or medians do not always exist; one of these spaces is a hyperplane of c_0 considered in [12]. (This does not contradict Theorem 2.1.) Examples of four-point sets with a center but without median, or with a median but without a center are indicated in [12,20]. Examples of three-point sets without k-centra for any k are shown at the end of this paper.

Remark 2.5. Let $A \subset F$, A containing at least k points, F finite. Then $r_k(A,x) \leqslant r_k(F,x)$ for all $x \in X$, and so $r_k(A) \leqslant r_k(F)$ $(1 \leqslant k \leqslant \#A)$. Also, if $r_k(A) = r_k(F)$, then $s_k(A) \subset s_k(F)$.

Remark 2.6. If $m_k \in s_k(A)$ and c is a center of A, then we have the almost trivial estimate

$$\|m_k - c\| \leqslant d(A, m_k) + r(A), \tag{6}$$

where $d(A, m_k) = \inf_{x \in A} \|x - m_k\|$ denotes the distance of m_k from the set A. In fact, if $\|m_k - a_i\| = d(A, m_k)$, then we have

$$\|m_k - c\| \leqslant \|m_k - a_i\| + \|a_i - c\| \leqslant d(A, m_k) + r(A).$$

Remark 2.7. It is clear that $x \in s_n(A)$ and $\|x - a_i\| =$ constant $i = 1, 2, \ldots, n$, implies $x \in s_1(A)$. (See, for example, [3] for results of this type.) More generally, if $c_k \in s_k(A)$ and the k farthest points to c_k in A are at the same distance r_k from c_k, then we have $r(A) \leqslant r(A, c_k) = r_k(A)$; so for $i = 1, \ldots, k$, $r_i(A) = r_k(A)$, and then $c_k \in s_i(A)$.

3. General results on *k*-centra

We start with a general result concerning k-centra, which generalizes results contained in [23], well-known for $k = \#A$.

Theorem 3.1. *Let X be a strictly convex space and $A \subset X$; if k is odd, then $s_k(A)$ $(1 \leqslant k \leqslant n)$ contains at most one point; if k is even and $s_k(A)$ contains x' and x'', $x' \neq x''$, then there exist (at least) k points of A on the line passing through x' and x''.*

Proof. Given $A = \{a_1, \ldots, a_n\}$ and k, $1 \leqslant k \leqslant n$, if x', x'' belong to $s_k(A)$, then according to the convexity of $s_k(A)$ also $x = (x' + x'')/2$ belongs to $s_k(A)$. Let $a_1, \ldots, a_k$ be the k points of A furthest away to x, so that $\sum_{i=1}^{k} \|x - a_i\| = k r_k(A)$. Then, we have

$$\begin{aligned} k r_k(A) &= \sum_{i=1}^{k} \left\| \frac{x' + x''}{2} - a_i \right\| \leqslant \sum_{i=1}^{k} \left(\frac{\|x' - a_i\|}{2} + \frac{\|x'' - a_i\|}{2} \right) \\ &\leqslant \frac{k r_k(A, x')}{2} + \frac{k r_k(A, x'')}{2} = k r_k(A), \end{aligned}$$

so all these inequalities are equalities. This means two facts: (1) $a_1, \ldots, a_k$ are also the k points in A furthest to x' and x''; and (2) $x' - a_i = \lambda_i (x'' - a_i)$ for some non-negative λ_i, $i = 1, \ldots, k$; therefore x', x'', $a_1, \ldots, a_k$ are all collinear. This is impossible for k odd because in this case the unique median of $A' = \{a_1, \ldots, a_k\}$ is the only point of A' leaving $(k-1)/2$ points of $a_1, \ldots, a_k$ to each side ("central point"); for k even, all points letting $k/2$ on each side are medians of A'. □

Remark 3.1. The proof of the above theorem shows that if X is a strictly convex space and $A \subset X$, if $\#A$ is odd, or $\#A$ is even and does not contain k collinear points, then $s_k(A)$ $(1 \leqslant k \leqslant n)$ contains at most one point. (The last result follows also from Proposition 2.2.)

When $k = 2$ we have no uniqueness result. (See Remark 3.3 below.)

Theorem 3.2. *For any $A \subset X$ we have $r(A) = r_2(A)$.*

Proof. Assume by contradiction, that $r_2(A) < r(A)$ for some $A = \{a_1, \dots, a_n\}$. Take $x \in X$ such that $r_2(A, x) = r(A) - \sigma$ for some $\sigma > 0$; we have $r(A, x) \geqslant r(A)$ (by definition) so there exists $a_i \in A$ such that $\|x - a_i\| \geqslant r(A)$.

For any $a_j \in A$, $j \neq i$, we have

$$\frac{\|x - a_i\| + \|x - a_j\|}{2} \leqslant r_2(A, x) = r(A) - \sigma,$$

so

$$\|x - a_j\| \leqslant 2r(A) - 2\sigma - \|x - a_i\| \leqslant 2r(A) - 2\sigma - r(A) = r(A) - 2\sigma.$$

If $x_\lambda = \lambda a_i + (1 - \lambda)x$, $0 \leqslant \lambda \leqslant 1$, then we have $\|x_\lambda - x\| = \lambda\|a_i - x\|$;

$$\begin{aligned}\frac{1}{2}\big(\|a_i - x_\lambda\| + \|x_\lambda - a_j\|\big) &\leqslant \frac{1}{2}\big(\|a_i - x\| - \|x - x_\lambda\| + \|x_\lambda - x\| + \|x - a_j\|\big)\\ &\leqslant r(A) - \sigma \quad \text{for all } j \neq i.\end{aligned}$$

Choose $\lambda \in (0, 1)$ so that $\|x_\lambda - a_i\| = r(A) - \sigma$; we obtain, for all $j \neq i$

$$\|x_\lambda - a_j\| \leqslant 2\big(r(A) - \sigma\big) - \|a_i - x_\lambda\| = 2r(A) - 2\sigma - \big(r(A) - \sigma\big) = r(A) - \sigma;$$

therefore $r(A, x_\lambda) \leqslant r(A) - \sigma$, a contradiction. □

Remark 3.2. In general, in any space, we have $r_3(A) < r_2(A)$ for some A: for example, also in the Euclidean plane E^2, there are three-point sets where the center and the median do not coincide.

We have proved (Theorem 3.2) that $r_1(A) = r_2(A)$ always. On the contrary, the equality $r_k(A) = r_{k+1}(A)$ for $k \geqslant 2$ does not happen frequently and it has some strong implications. We shall discuss now this fact, giving a converse of Remark 2.7.

Theorem 3.3. *Let $r_k(A) = r_{k+1}(A)$ for some $k \geqslant 1$ and $A = \{a_1, \dots, a_n\}$; $n > k$. Then $s_k(A) \subset s_{k+1}(A)$. (In particular, by Theorem 3.2, if c is a center of A, then $c \in s_2(A)$.) Moreover, if $c_k \in s_k(A)$, then (at least) the $k + 1$ points of A which are farthest to c_k have the same distance $r_k(A)$ from it; in addition, for $i = 1, \dots, k$, $r_i(A) = r_k(A)$; $c_k \in s_i(A)$; $s_i(A) \subset s_{i+1}(A)$. (Note that if X is strictly convex, then $s_{k+1}(A)$ is a singleton for $k \geqslant 2$ since the $k + 1$ points farthest to c_k are not collinear.)*

Proof. Let $r_k(A) = r_{k+1}(A)$; $c_k \in s_k(A)$. Order the elements of A so that $\|c_k - a_1\| \geqslant \|c_k - a_2\| \geqslant \dots \geqslant \|c_k - a_n\|$; we have

$$r_k(A) = \frac{1}{k}\sum_{i=1}^{k}\|c_k - a_i\| \geqslant \frac{1}{k+1}\sum_{i=1}^{k+1}\|c_k - a_i\| = r_{k+1}(A, c_k) \geqslant r_{k+1}(A).$$

Therefore, our assumption implies that $c_k \in s_{k+1}(A)$; moreover,

$$\frac{1}{k+1}\left(\sum_{i=1}^{k}\|c_k - a_i\| + \|c_k - a_{k+1}\|\right) = \frac{1}{k}\sum_{i=1}^{k}\|c_k - a_i\|$$

implies

$$\frac{\|c_k - a_{k+1}\|}{k+1} = \left(\frac{1}{k} - \frac{1}{k+1}\right)\sum_{i=1}^{k}\|c_k - a_i\| = \frac{r_k(A)}{k+1},$$

so $\|c_k - a_{k+1}\| = r_k(A)$; but then, since

$$\|c_k - a_{k+1}\| \leqslant \min_{1\leqslant i\leqslant k}\|c_k - a_i\| \leqslant \frac{1}{k}\sum_{i=1}^{k}\|c_k - a_i\| = r_k(A),$$

$\|c_k - a_1\| = \cdots = \|c_k - a_k\| = \|c_k - a_{k+1}\|$. By recalling Remark 2.7, we obtain the conclusion. □

Remark 3.3. In general, also if X is the Euclidean plane, a 2-centrum of A is not a center: for example, if $A = \{(0,1); (0,-1); (\varepsilon,0)\}$, $0 \leqslant \varepsilon \leqslant 1$, then the unique center of A is the origin, while all points $(0,\alpha)$; $|\alpha| \leqslant (1-\varepsilon^2)/2$, are 2-centra.

Remark 3.4. If A has at most one $(k+1)$-centrum and $r_k(A) = r_{k+1}(A)$, then $x \in s_{k+1}(A) \Rightarrow s_k(A) \subseteq \{x\}$. Without the assumption of uniqueness on $s_{k+1}(A)$ this is not true, as the following example shows. Let X be the plane with the max norm, and $A = \{(-\frac{9}{10},0); (\frac{11}{10},1); (-\frac{9}{10},-1)\}$; we have $r_2(A) = r_3(A) = 1$; $P = (\frac{1}{10},0)$ belongs to $s_2(A) \subset s_3(A)$; the origin belongs to $s_3(A)$ but not to $s_2(A)$.

Our next result, whose proof follows from the definition of $r_k(A)$, extends [3, Proposition 2.7].

Theorem 3.4. *Let $m_k \in s_k(A)$, $m_j \in s_j(A)$, $\max\{k,j\} \leqslant n = \#A$. Then we have*

$$\|m_k - m_j\| \leqslant r_k(A) + r_j(A). \tag{7}$$

In particular, if $j = k$ and $\{m_k, m_k'\} \subset s_k(A)$, then

$$\|m_k - m_k'\| \leqslant 2r_k(A). \tag{8}$$

Remark 3.5. The estimates (7) and (8) are sharp. (See [3, Example 2.9].) But if we assume that X is strictly convex, then we have better estimates. In fact, according to Remark 3.1, in this case (for $k \neq 2$) we have uniqueness of solutions in many cases. But for $k \neq j$ we cannot give better inequalities (see [4, §4]) apart from the fact that strict inequality holds in both (7) and (8).

Now assume that we have equality in (7). In proving Theorem 3.4, we obtain subsequently; for the j farthest points to m_j, a_i, $i = 1,2,\ldots,j$, we have $\|m_j - a_i\| + \|a_i - m_k\| = \|m_j - m_k\|$; the j farthest points to m_k, all have distance $r_k(A,m_k)$ from it; therefore, if $j > k$ then $r_k(A) = r_j(A)$ and both m_k and m_j belong to $s_j(A)$. If $j = k$, then the k farthest points to m_j $[m_k]$ are on the sphere of radius r_k centered at m_j [respectively at m_k]; moreover the distance between the centers of the two balls is twice the radius r_k.

In the following we consider a localization property of the k-centra with respect to $\operatorname{co}(A)$, the convex hull of the set A.

Theorem 3.5. *If X is a two-dimensional space, or if X is a Hilbert space, then for any A and any k $(1 \leqslant k \leqslant \#A)$, it holds $s_k(A) \cap \operatorname{co}(A) \neq \emptyset$. Moreover, if X is a Hilbert space, or if $\dim(X) = 2$ and X is strictly convex, then $s_k(A) \subset \operatorname{co}(A)$.*

Proof. The assumptions imply that $s_k(A) \neq \emptyset$. If $\dim(X) = 2$ then (see [21]) for every $x \in X$ there exists $x^* \in \operatorname{co}(A)$ such that $\|x^* - a\| \leqslant \|x - a\|$ for any $a \in A$; i.e., $\|x^* - a_i\| \leqslant \|x - a_i\|$ for $i = 1, \ldots, n = \#A$, so $r_k(A, x^*) \leqslant r_k(A, x)$: if we take $x \in s_k(A)$, this shows that there also exists $x^* \in s_k(A) \cap \operatorname{co}(A)$.

Now let X be Hilbert or if $\dim(X) = 2$, X strictly convex; if $x \notin \operatorname{co}(A)$, let x^* be the best approximation to x from $\operatorname{co}(A)$: we have $\|x^* - a_i\| < \|x - a_i\|$ for $i = 1, \ldots, n$, so $r_k(A, x^*) < r_k(A, x)$, thus an element of $s_k(A)$ must belong to $\operatorname{co}(A)$. □

Corollary 3.1. *Let X be Hilbert or if $\dim(X) = 2$, X strictly convex; given $A \subset X$ with no subset of k points being collinear, if $m_k \in s_k(A)$ and $c \in s_1(A)$, then $\|m_k - c\| = r(A)$ implies that $m_k \in A$.*

Proof. Follow the line of the proof of [4, Proposition 5.1]. □

Another interesting property of k-centra of a set A is that they allow to characterize inner product spaces in terms of their intersection with the convex hull of A. Characterizations of this type are known from the sixties. (See [8,9].) The same property concerning medians was considered in the nineties by Durier [7], where partial answers were given. It has been proved only recently for medians of three-point sets, this result can be found in [6].

Theorem 3.6. *If $\dim(X) \geqslant 3$ and the norm of X is not Hilbertian, then there exists a three-point set A such that $s_3(A) \cap \operatorname{co}(A) = \emptyset$.*

By using such theorem, it is not difficult to obtain the following proposition.

Proposition 3.1. *If $\dim(X) \geqslant 3$ and the norm of X is not Hilbertian, then for every $n \geqslant 3$ there exists an n-point set F such that $s_3(F) \cap \operatorname{co}(F) = \emptyset$.*

Proof. We prove the result for $n = 4$, the extension to $n \geqslant 4$ being similar.

Under the assumptions done, according to Proposition 2.2, $\inf_{x \in \operatorname{co}(A)} r_3(A, x)$ is always attained; now take $A = \{a_1, a_2, a_3\}$ as given by Theorem 3.6: for some $\sigma > 0$ we have

$$\inf_{x \in \operatorname{co}(A)} r_3(A, x) = r_3(A) + 4\sigma > r_3(A).$$

Take $\bar{x} \in X$ such that $r_3(A, \bar{x}) < r_3(A) + \sigma$; it is not a restriction to assume that $\|\bar{x} - a_3\| \leqslant \min\{\|\bar{x} - a_1\|, \|\bar{x} - a_2\|\}$. Now take $a_4 \notin A$ such that $\|a_3 - a_4\| \leqslant \sigma$ and let $F = A \cup \{a_4\}$. We have $r_3(F, \bar{x}) \leqslant r_3(A, \bar{x}) + \sigma \leqslant r_3(A) + 2\sigma$. Now take $y \in \mathrm{co}(F)$: there is $x \in \mathrm{co}(A)$ such that $\|x - y\| \leqslant \sigma$; therefore $|r_3(F, y) - r_3(F, x)| \leqslant \sigma$, so $r_3(F, y) \geqslant r_3(F, x) - \sigma \geqslant r_3(A, x) - \sigma \geqslant r_3(A) + 3\sigma$; thus

$$\inf_{y \in \mathrm{co}(F)} r_3(F, y) \geqslant r_3(A) + 3\sigma \geqslant r_3(F, \bar{x}) + \sigma \geqslant r_3(F) + \sigma,$$

this proves the thesis. □

Given a set A with n points and $k < n$, we can divide the space X into $\binom{n}{k}$ regions R_j, so that when x is taken in one of these regions, the same k points of A are the farthest to x; of course, inside each of these regions there are $k!$ different possible orderings $\sigma_1, \ldots, \sigma_k$. It is possible to have $R_i \cap R_j \neq \emptyset$ (the values of the kth distance can be equal to the $(k+1)$th one); also, if R_j is determined by $a_1, \ldots, a_k$ then $a_i \notin R_j$ for $i = 1, \ldots, k$. Also in general the medians of $a_1, \ldots, a_k$ (if they exist) do not belong to R_j. Note that these regions are not in general convex: for example, if X if the plane with the max norm, given $a_1 = (1, 0)$ and $a_2 = (-1, 0)$, the set $\|x - a_1\| \geqslant \|x - a_2\|$ is not convex. But the same is true, for some pair, in any space with a non-hilbertian norm.

If X is a Hilbert space, then the regions R_j are convex: in fact, consider, e.g., the region R determined by the points $a_1, \ldots, a_k$, $k < \#A$: then

$$R = \bigcap_{i=1}^{k} \bigl\{x \in X\colon \ \|x - a_h\| \leqslant \|x - a_i\| \text{ for } h = k+1, \ldots, n\bigr\}.$$

R is the intersection of $k(n-k)$-convex regions, therefore it is convex. A detailed analysis of these sets can be found in [13]. (Not only for Hilbert spaces.) Also in the particular case of two-dimensional spaces some geometrical properties as well as the complexity analysis are given in [14].

Minimizing $r_k(A)$ is equivalent to solve $\binom{n}{k}$ constrained Fermat problems; then looking for the minimum of the values obtained: for each R_j, determined by k given points, say $\{a_1, \ldots, a_k\}$, look for a median of these points, restricted to the "feasible region" R_j. Algorithms for the solution of this kind of problems in two-dimensional spaces can be found in [14]; also, in networks (finite metric spaces) algorithms are given in [10,16].

Given X, consider for $k \in \mathbb{N}$ the parameter

$$J_k(X) = \sup\left\{\frac{2r_k(A)}{\delta(A)}\colon \ A \subset X \text{ finite, } \max\{2, k\} \leqslant \#A\right\}. \tag{9}$$

For $k = 1$, the number $J_1(X) = J(X)$ is called the finite Jung constant and has been studied intensively; in general, $1 \leqslant J(X) \leqslant 2$, while the value of $J(X)$ gives information on the structure of X. As shown partially in [5] and later completely in [18], we always have

$$J(X) = \sup\left\{\frac{2\mu(A)}{\delta(A)}\colon \ A \subset X \text{ finite, } 2 \leqslant n = \#A\right\}.$$

Since $\mu(A) \leqslant r_k(A) \leqslant r(A)$ always (see (1)), we obtain the following result.

Theorem 3.7. *In every space X, for every positive integer k, we have*

$$J_k(X) = J(X). \tag{10}$$

Our last result in this section was already known for medians (see [4]) but it can be extended to general k-centra.

Proposition 3.2. *Let $m_k \in s_k(A)$ for some set A. Assume that $A_k \subset A$, $\#A_k = k$ and $r_k(A) = \frac{1}{k}\sum_{a\in A_k}\|m_k - a\|$. If $\|m_k - \frac{1}{k}\sum_{a\in A_k} a\| = r_k(A)$ then X is not strictly convex.*

Proof. By the triangular inequality we have

$$r_k(A) = \left\| m_k - \frac{1}{k}\sum_{a\in A_k} a \right\| \leqslant \frac{1}{k}\sum_{a\in A_k}\|m_k - a\| = r_k(A).$$

Thus, m_k is also a center of A_k and $r_k(A) = r(A_k)$. Now, we apply first claim in [4, Proposition 3.1] to the set A_k to get the result. □

4. Concluding remarks

To conclude our analysis of k-centra, we study several properties of these points regarding equilateral sets. Recall that A is called *equilateral* if $\|a_i - a_j\| =$ constant for $i \neq j$, $1 \leqslant i, j \leqslant n = \#A$. Also, recall that the centroid of a finite set A is given by the point $\frac{1}{\#A}\sum_{a\in A} a$. For equilateral sets there are several nice properties connecting centers, medians and centroids (see [2]). Some of them can be extended further to k-centra.

Proposition 4.1. *Let A be an equilateral set in a Hilbert space X and let $k \geqslant 3$; then the centroid of A belongs to $s_k(A)$.*

Proof. Assume that 0 is the center of A; then $\langle a_i, a_j\rangle =$ constant for $i \neq j$, $1 \leqslant i, j \leqslant n = \#A$. Let $y = \sum_{j=1}^n \lambda_j a_j$; then the function $f(\lambda_1, \dots, \lambda_n) = \sum_{i=1}^k \|y - a_i\|$ is symmetric.

In Hilbert spaces it always exists $m_k \in s_k(A) \cap \mathrm{co}(A)$. Moreover, under the hypothesis of the proposition $s_k(A)$ is a singleton, then m_k is the unique minimizer of f and $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1/n$; thus m_k is the centroid of A. □

Remark 4.1. Let $A = \{a_1, \dots, a_n\}$ be an equilateral set with $\|a_i - a_j\| = d$, $\forall i \neq j$; then it is easy to see that

$$r_k(A, x) \geqslant \frac{d}{2} \quad \text{for any } x \in X.$$

Indeed, for any $x \in X$, $kr_k(A, x)$ is attained as a sum of distances from x to k points of A. Let us denote by $A_k(x)$ the subset of A containing the points that define $r_k(A, x)$. $A_k(x)$ itself is an equilateral set with $\|a_i - a_j\| = d$, $\forall i \neq j$, $a_i, a_j \in A_k(x)$; then

$$kr_k(A, x) = \sum_{a\in A_k(x)} \|a - x\| \geqslant \frac{kd}{2},$$

where inequality comes from [2, Lemma 4.1] applied to the set $A_k(x)$. (Also if k is even, it follows from (3).)

Proposition 4.2. *For any equilateral set in the hypothesis of Remark* 4.1*, the conditions* $r(A) = d/2$ *and* $r_k(A) = d/2$ *are equivalent, for any* $k = 2, 3, \ldots, n$*. In these cases* k*-centra for any* $k = 1, 2, \ldots, n = \#A$ *coincide.*

Proof. Runs parallel to [2, Proposition 5.1] except for the details of considering partial sums of k-largest distances. □

From this last result we can present an example of set without k-centra for any k. [2, Example 5.2] is an equilateral three-point set without median. Now, we apply Proposition 4.2 to conclude that the set in that example cannot have k-centra for any $k = 1, 2, 3$.

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